



Alternative Approach for the Derivation of Black-Scholes Partial Differential Equation in the Theory of Options Pricing Using Risk Neutral Binomial Process

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Authors' contributions

This work was carried out in collaboration between all authors. Author SEF designed the study, wrote the protocol, and wrote the first draft of the manuscript. Author JTO managed the literature searches and numerical implementation of binomial model for vanilla options pricing. Author EIA managed numerical experiment and conclusion. All authors read and approved the final manuscript.

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ABSTRACT

This paper presents a risk neutral binomial process as an alternative approach for the derivation of analytic pricing equation called "Black-Scholes Partial differential Equation" in the theory of option pricing.

Binomial option pricing is a powerful technique that can be used to solve many complex option-pricing problems. In contrast to the Black-Scholes model and other option pricing models that require solutions to stochastic differential equations, the binomial model is mathematically simple. Binomial model is based on the assumption of no arbitrage. The assumption of no arbitrage implies that all risk-free investments earn the risk-free rate of return and no investment opportunity exists that requires zero amounts of investment but yield positive returns.

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We derive Black-Scholes partial differential equation using risk neutral binomial process. We also discuss the convergence of binomial model to the analytic pricing formula, the Black-Scholes model for pricing options.

Binomial model has the Black-Scholes analytic formula as the limiting case as the number of steps tends to infinity. This model is much more capable of handling options with early exercise because it considers the cash flow at each time period rather than just the cash flows at expiration.

Keywords: American option; binomial model; black-scholes model; black-scholes partial differential equation; convergence; european option; vanilla option.

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1. INTRODUCTION

Binomial model is defined as an iterative approach that models the price evolution over the whole option validity period. For some vanilla options such as American option, binomial model is the only choice since there is no known closed form solution that predicts its price over a period of time.

Black-Scholes model [1] seems dominated the option pricing, but it is not the only popular model, the Cox-Ross-Rubinstein "Binomial" model is also popular. The Cox-Ross-Rubinstein "Binomial" model has the Black-Scholes analytic formula as the limiting case as the number of steps tends to infinity.

Cox-Ross-Rubinstein [2] presented the binomial tree model in the paper titled "Option Pricing: A simplified approach" in 1979. This model is relatively simple and easy to understand, but it is an extremely powerful tool for pricing a wide range of options. They found a better stock movement model other than the geometric Brownian motion model applied by Black and Scholes, the binomial tree model. The tree specifies precisely all the possible future stock prices and the associated possibilities to obtain those prices.

The rate of return on the stock over each period can have two possible values: u with possibility q , or d with probability $(1 - q)$. Thus, if the current underlying price of the asset is S , the stock price at the end of the period will be either Su or Sd . The binomial model of the stock price Expanding $V(Su, \Delta t)$ in Taylor series yields

movements is a discrete time model as opposes to the geometric Brownian motion model, which is a continuous time model.

We present an overview of binomial model in the context of Black-Scholes-Merton for pricing vanilla options based on the risk-neutral valuation which was first suggested and derived by [2] and assumes that stock price movements consist of a large number of small binomial movements. For more literatures on the theory of options pricing see [3-10] just to mention a few.

In this paper we shall consider the binomial model as an alternative method for deriving Black-Scholes partial differential equation and its convergence to the Black-Scholes model.

2. AN ALTERNATIVE APPROACH FOR DERIVING BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION

In this section, we will show that given the risk neutral binomial process, we can derive the Black-Scholes equation from the risk-neutral expectation formula given below:

$$V_0 = e^{-rT} E_q[f] \quad (1)$$

Let us consider one period binomial model. Let the current spot underlying price of the asset be denoted by S . The risk neutral expectation is given below:

$$E_q[V(S, 0)] = qV(Su, \Delta t) + (1 - q)V(Sd, \Delta t) \quad (2)$$

$$\begin{aligned} V(Su, \Delta t) &= V(S + S(u - 1), \Delta t) \\ &= V(S, \Delta t) + V'(S, \Delta t)S(u - 1) + \frac{1}{2}V''(S, \Delta t)S^2(u - 1)^2 + 0(u^3) \end{aligned} \quad (3)$$

Similarly we expand $V(Sd, \Delta t)$ in Taylor series:

$$\begin{aligned} V(Sd, \Delta t) &= V(S + S(d - 1), \Delta t) \\ &= V(S, \Delta t) + V'(S, \Delta t)S(d - 1) + \frac{1}{2}V''(S, \Delta t)S^2(d - 1)^2 + 0(d^3) \end{aligned} \quad (4)$$

Substituting (3) and (4) into (5) we have that:

$$\begin{aligned} E_q[V(S, 0)] &= q \left[V(S, \Delta t) + V'(S, \Delta t)S(u - 1) + \frac{1}{2}V''(S, \Delta t)S^2(u - 1)^2 + 0(u^3) \right] + (1 - q) \left[V(S, \Delta t) + V'(S, \Delta t)S(d - 1) + \frac{1}{2}V''(S, \Delta t)S^2(d - 1)^2 + 0(d^3) \right] \\ &= q \left[V(S, \Delta t) + SuV'(S, \Delta t) - SV'(S, \Delta t) + \frac{1}{2}u^2V''(S, \Delta t)S^2 + \frac{1}{2}V''(S, \Delta t)S^2 - uV''(S, \Delta t)S^2 \right] \\ &\quad + (1 - q) \left[V(S, \Delta t) + V'(S, \Delta t)dS - SV'(S, \Delta t) + \frac{1}{2}d^2V''(S, \Delta t)S^2 + \frac{1}{2}V''(S, \Delta t)S^2 - dV''(S, \Delta t)S^2 \right] \\ &= qV(S, \Delta t) + SuV'(S, \Delta t)q - SV'(S, \Delta t)q + \frac{1}{2}u^2V''(S, \Delta t)S^2 + \frac{1}{2}V''(S, \Delta t)S^2 - uV''(S, \Delta t)S^2 + \\ &\quad V(S, \Delta t) + V'(S, \Delta t)dS - SV'(S, \Delta t) + \frac{1}{2}d^2V''(S, \Delta t)S^2 + \frac{1}{2}V''(S, \Delta t)S^2 - dV''(S, \Delta t)S^2 - qV(S, \Delta t) - \\ &\quad qdV'(S, \Delta t)S + qV'(S, \Delta t)S - \frac{1}{2}qd^2V''(S, \Delta t)S^2 - \frac{1}{2}qV''(S, \Delta t)S^2 + qdV''(S, \Delta t)S^2 \\ &= V(S, \Delta t) + V'(S, \Delta t)S[q(u - 1) - (1 - q)(d - 1)] + \frac{1}{2}V''(S, \Delta t)S^2(u - 1)^2 + 0(u^3) \end{aligned} \quad (5)$$

Recall that

$$qu + (1 - q)d = e^{r\Delta t} \quad (6)$$

and by risk neutral argument, we have that:

$$V(S, 0)e^{r\Delta t} = V(S, 0)(1 + r\Delta t) + 0(\Delta t^2) \quad (7)$$

Simplifying (5) further and substituting (6) into (5), then (5) becomes;

$$\begin{aligned} E_q[V(S, 0)] &= V(S, \Delta t) + V'(S, \Delta t)S[e^{r\Delta t} - 1] \\ &\quad + \frac{1}{2}V''(S, \Delta t)S^2(u - 1)^2 + 0(u^3) \\ &= V(S, \Delta t) + V'(S, \Delta t)Sr\Delta t + \\ &\quad \frac{1}{2}V''(S, \Delta t)S^2\sigma^2\Delta t + 0(\Delta t^3) \end{aligned} \quad (8)$$

Since $E_q[V(S, 0)] = V(S, 0)e^{r\Delta t}$, by comparing (7) and (8) we have that;

$$\begin{aligned} V(S, 0)e^{r\Delta t} &= V(S, \Delta t) + V'(S, \Delta t)S[e^{r\Delta t} - 1] \\ &\quad + \frac{1}{2}V''(S, \Delta t)S^2(u - 1)^2 + 0(u^3) \end{aligned}$$

Solving further the last equation and rearranging we have

$$\begin{aligned} V(S, \Delta t) - V(S, 0) + \frac{1}{2}S^2\sigma^2V''(S, \Delta t) + \\ rSV'(S, \Delta t)\Delta t - rV(S, 0)\Delta t + 0(\Delta t^3) = 0 \end{aligned} \quad (9)$$

Taking the limit of (9) as $\Delta t \rightarrow 0$, we get the Black-Scholes partial differential equation given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad (10)$$

2.1 Binomial Model

We show the parameters of binomial model for continuous time prices using the lognormal price process. Consider the binomial parameters which are defined below as:

$$u = e^{(r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} \quad (11)$$

$$d = e^{(r - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}} \quad (12)$$

and

$$q = \frac{e^{r\Delta t} - d}{u - d} \quad (13)$$

which are not the only possible ways to construct a risk neutral binomial tree. The lognormal model is fully specified by the mean and variance of the random variable,

$$S_T = S_0e^{X_T} \text{ or } e^{X_T} = \frac{S_T}{S_0} \quad (14)$$

Where

$$X_t = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t \quad (15) \quad \text{Comparing (20) and (22) we have}$$

The variance of e^{X_T} is:

$$\text{var}[e^{X_T}] = E[(e^{X_T})^2] - (E[e^{X_T}])^2 = E[e^{2X_T}] - (E[e^{X_T}])^2 \quad (16)$$

where e^{X_T} has a mean of e^{rT} . To find the mean e^{2X_T} , we will apply Ito's calculus to (15) given by:

$$X_t = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t, \quad \text{this implies that}$$

$$2X_t = 2\left(r - \frac{1}{2}\sigma^2\right)t + 2\sigma W_t$$

Therefore,

$$dX_t = 2\left(r - \frac{1}{2}\sigma^2\right)dt + \frac{1}{2}(2\sigma)^2dt + 2\sigma dW_t$$

$$dX_t = 2r dt - \sigma^2 dt + \frac{1}{2}(4\sigma^2)dt + 2\sigma dW_t$$

$$dX_t = 2r dt - \sigma^2 dt + 2\sigma^2 dt + 2\sigma dW_t$$

$$dX_t = (2r + \sigma^2)dt + 2\sigma dW_t \quad (17)$$

This implies that in the risk neutral world the stochastic differential equation is $dX_t = \mu dt + \sigma dW_t$ has drift $\mu = (2r + \sigma^2)$.

From (17), e^{2X_T} has mean $e^{(2r+\sigma^2)T}$ and the variance of e^{X_T} is given below:

$$\text{var}[e^{X_T}] = e^{2rT+\sigma^2T} - (E[e^{X_T}])^2 \quad (18)$$

Since,

$$e^{X_T} = \frac{S_T}{S_0} \quad (19)$$

The mean and the variance of (19) are defined respectively as

$$E\left[\frac{S_T}{S_0}\right] = e^{rT} \quad (20)$$

$$\text{var}\left[\frac{S_T}{S_0}\right] = e^{(2r+\sigma^2)T} - \left(E\left[\frac{S_T}{S_0}\right]\right)^2 \quad (21)$$

We will apply mean and variance to one period binomial model with $T = \Delta t$ and taking the limit as $\Delta t \rightarrow 0$. In this model, the mean and variance of (19) can also be written as given below:

$$E\left[\frac{S_T}{S_0}\right] = qu + (1 - q)d \quad (22)$$

$$\text{var}\left[\frac{S_T}{S_0}\right] = qu^2 + (1 - q)d^2 - E\left[\frac{S_T}{S_0}\right]^2 \quad (23)$$

$$E\left[\frac{S_T}{S_0}\right] = E\left[\frac{S_T}{S_0}\right] \text{ i.e. } e^{rT} = qu + (1 - q)d, \text{ for}$$

$T = \Delta t$, we have that

$qu + (1 - q)d = e^{r\Delta t}$
Similarly by comparing (21) and (23) we deduce that $\text{var}\left[\frac{S_T}{S_0}\right] = \text{var}\left[\frac{S_T}{S_0}\right]$ i.e.

$$qu^2 + (1 - q)d^2 - \left(E\left[\frac{S_T}{S_0}\right]\right)^2$$

$$= e^{(2r+\sigma^2)T} - \left(E\left[\frac{S_T}{S_0}\right]\right)^2,$$

$qu^2 + (1 - q)d^2 = e^{(2r+\sigma^2)T}$, using the fact that $T = \Delta t$, then we have:

$$qu^2 + (1 - q)d^2 = e^{(2r+\sigma^2)\Delta T}$$

$$qu^2 + (1 - q)d^2 = e^{2r\Delta t + \sigma^2\Delta t}$$

Hence we have two equations given by:

$$qu + (1 - q)d = e^{r\Delta T} \quad (24)$$

$$qu^2 + (1 - q)d^2 = e^{2r\Delta t + \sigma^2\Delta t} \quad (25)$$

Equations (24) and (25) have three unknown variables. Let us set $q = \frac{1}{2}$

Substituting the value of $q = \frac{1}{2}$ into (24) and (25), we have that

$$(u + d) = 2e^{r\Delta t} \quad (26)$$

$$(u^2 + d^2) = 2e^{2r\Delta t + \sigma^2\Delta t} \quad (27)$$

Solving (26) and (27) yields

$$u = e^{r\Delta t} \left(1 + \sqrt{e^{\sigma^2\Delta t} - 1}\right) \quad (28)$$

$$d = e^{r\Delta t} \left(1 - \sqrt{e^{\sigma^2\Delta t} - 1}\right) \quad (29)$$

$$q = \frac{1}{2} \quad (30)$$

This proves that the binomial model approximates the lognormal price process.

Next to conclude the decision about the equivalence of binomial and Black-Scholes

models we consider the convergence of the binomial model to the Black-Scholes model.

2.2 Convergence of Binomial Model to the Black-Scholes Model

The Black-Scholes formula for the price of a European call option is

$$C_{T|0} = S_0 \Phi(d_1) - K_{T|0} e^{-rT} \Phi(d_2) \quad (31)$$

where $\Phi(d)$ denotes the value of the cumulative Normal distribution function i.e the probability that $Z \leq d$ when $Z \sim N(0,1)$ is a standard normal variable and where

$$d_1 = \frac{\ln(S_0/K_{T|0}) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad (32a)$$

$$d_2 = \frac{\ln(S_0/K_{T|0}) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \quad (32b)$$

Now, we want to show that as the number of the subintervals “ n ” of the finite period $[0, T]$ increases indefinitely, the binomial formula for the value $C_{T|0}$ of the call option converges to Black-Scholes formula. We begin by simplifying the binomial formula. Observe that for some outcomes there is $\max(S_0 u^j d^{n-j} - K_{T|0}) = 0$. Let a be the smallest number of upward movements of the underlying stock price that will ensure that the call option has a positive value, which is to say that it finishes in-the-money. Then $S_0 u^a d^{n-a} = K_{T|0}$; and only the binomial paths from $j = a$ onwards needs to be taken into account. Therefore the equation for the generalization n -sub periods is given by:

$$C_{T|0} = e^{-rT} \left\{ \sum_{j=0}^n \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} C^{uj,d(n-j)} \right\} = e^{-rT} E(C_{T|T}) \quad (33)$$

where $C^{uj,d(n-j)} = \max[S_0 u^j d^{n-j} - K_{T|0}] = [S_0 u^j d^{n-j} - K_{T|0}]^+$

Therefore equation (33) becomes

$$C_{T|0} = e^{-rT} \left\{ \sum_{j=a}^n \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} [S_0 u^j d^{n-j} - K_{T|0}] \right\} \quad (34)$$

$$C_{T|0} = \left\{ e^{-rT} \sum_{j=a}^n \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} S_0 u^j d^{n-j} \right\} - K_{T|0} e^{-rT} \left\{ \sum_{j=a}^n \frac{n!}{(n-j)!j!} p^j (1-p)^{n-j} \right\} \quad (35)$$

To demonstrate that this converges to equation (31) as $n \rightarrow \infty$, it must be shown that the terms inside brace bracket in (35) associated with S_0 and $K_{T|0} e^{-rT}$, converge to $\Phi(d_1)$ and $\Phi(d_2)$ respectively.

The term associated with $K_{T|0} e^{-rT}$ is a binomial sum; and taking the limit as $n \rightarrow \infty$, it converges to the partial integral of a standard normal distribution. The term associated with S_0 can be simplified so that it becomes a binomial sum that converges to a normal integral. Define the growth factor R by the equation $R^n = e^{rT}$, then in reference to the equation that:

$$P = \frac{e^{rT} - d}{u - d} \text{ and } 1 - P = \frac{u - e^{rT}}{u - d}$$

It can be seen that within the context of the *one*-period binomial model, there is

$$P = \frac{R - d}{u - d} \text{ and } 1 - P = \frac{u - R}{u - d}$$

Now define: $P_* = \frac{u}{Rp}$ and $1 - P_* = \frac{d(1-p)}{R}$

Then the term associated with S_0 can be written as

$$\sum_{j=a}^n \frac{n!}{(n-j)!j!} P_*^j (1 - P_*)^{n-j} \quad (36)$$

The task is now to replace the binomial sums as $n \rightarrow \infty$ by corresponding partial integrals of the standard normal distribution.

Let us first observe that the condition $S_0 u^a d^{n-a} = K_{T|0}$ can be solved to give

$$a = \frac{\ln(K_{T|0}/S_0) - n \ln d}{\ln(u/d)} + O(n^{-1/2}) \quad (37)$$

Next let $S_T = S_0 u^j d^{n-j}$ be the stock price on expiry.

Therefore,

$$\ln(S_T/S_0) = \ln u^j d^{n-j} = j \ln u + (n-j) \ln d = j \ln(u/d) + n \ln d \quad (38)$$

Taking the expected value of (38) yields

$$E\{\ln(S_T|S_0)\} = E(j) \ln(u/d) + n \ln d \quad (39)$$

and

$$V\{\ln(S_T|S_0)\} = V(j)\{\ln(u/d)\}^2 \quad (40)$$

Solving (39) and (40) we have respectively as

$$E(j) = \frac{\{\ln(S_T|S_0)\} - n \ln d}{\ln(u/d)} \quad (41)$$

and

$$V(j) = \frac{V\{\ln(S_T|S_0)\}}{\{\ln(u/d)\}^2} \quad (42)$$

Now, the parameter, a which marks the first term in each of the binomial sums must be converted to a value that will serve as the limit of the corresponding integrals of the standard normal distribution.

The standardized value in question is

$$d = -\{a - E(j)\}/\sqrt{V(j)} \quad (43)$$

to which a negative sign has been applied to ensure that the integral is over the interval $(-\infty, d]$ which accords with the usual tabulation of the cumulative normal distribution instead of the interval to $(-d, \infty]$, which would correspond more directly to the binomial summation from a to n . Therefore by substituting (37), (41) and (42) into (43), then d gives

$$d = \frac{-\{a - E(j)\}}{\sqrt{V(j)}} = \frac{\ln(S_0|K_{T|0}) + E\{\ln(S_T|S_0)\}}{\sqrt{V\{\ln(S_T|S_0)\}}} - O(n^{-1/2}) \quad (44)$$

As $n \rightarrow \infty$, the term of order $n^{-1/2}$ vanishes. Also the trajectory of the stock price converges to a geometric Brownian and from continuous stochastic processes, we can gather the result that $V\{\ln(S_T|S_0)\} = \sigma^2 T$. This is irrespective of the size of the drift parameter μ , which will vary with the values p and p_* . Therefore we have

$$d = \frac{\ln(S_0|K_{T|0}) + E\{\ln(S_T|S_0)\}}{\sigma\sqrt{T}} \quad (45)$$

Next, we want to show that

$$E\{\ln(S_T|S_0)\} = \begin{cases} (r - \sigma^2/2)T & \text{if the probability of } u \text{ is } p \\ (r + \sigma^2/2)T & \text{if the probability of } u \text{ is } p_* \end{cases} \quad (46)$$

First, we consider $S_T|S_0 = \prod_{i=1}^n (S_i/S_{i-1})$, where S_n is the same as S_T . Since this is a product of sequence of independent and identically distributed random variables, there is

$$E(S_T|S_0) = \prod_{i=1}^n (S_i/S_{i-1}) = \{E(S_i/S_{i-1})\}^n \quad (47)$$

Moreover, since $S_i/S_{i-1} = u$ with probability p and $S_i/S_{i-1} = d$ with probability $(1-p)$, the expected value of this ratio is

$$E(S_i/S_{i-1}) = pu + (1-p)d = R \quad (48)$$

where the second equality follows in view of the definition of (36). Putting this back into (47) gives

$$E(S_T|S_0) = R^n \text{ and } \ln\{E(S_T|S_0)\} = n \ln R \quad (49)$$

It follows that from the property of the lognormal distribution

$$\ln\{E(S_T|S_0)\} = E\left\{\ln(S_T|S_0)\right\} + \frac{1}{2}V\{\ln(S_T|S_0)\} \quad (50)$$

Rearranging (50) yields

$$E\{\ln(S_T|S_0)\} = \ln\{E(S_T|S_0)\} - \frac{1}{2}V\{\ln(S_T|S_0)\} = (r - \sigma^2/2)T \quad (51)$$

The final equality follows from the definitions given by $R^n = e^{rT}$ and that $V\{\ln(S_T|S_0)\} = T\sigma^2$. This provides the first equality of (46)

Now in pursuit of the second equality of (46) we must consider $(S_0|S_T) = \prod_{i=1}^n (S_{i-1}/S_i)$ i.e. the inverse of the ratio in question is. In the manner of (47), there is

$$E(S_0|S_T) = \prod_{i=1}^n (S_{i-1}/S_i) = \{E(S_{i-1}/S_i)\}^n \quad (52)$$

However the expected value of the inverse ratio is

$$E(S_{i-1}/S_i) = p_*u^{-1} + (1-p_*)d^{-1} = R^{-1} \quad (53)$$

which follows in view of the definitions of P_* and $(1 - p_*)$ given by (37). Substituting these into (52) gives

$$E(S_0|S_T) = R^{-n} \text{ whence } \ln\{E(S_0|S_T)\} = n \ln R \quad (54)$$

From the property of the log normal distribution, (50) becomes

$$\begin{aligned} \ln\{E(S_0|S_T)\} &= E\left\{\ln(S_0|S_T)\right\} + \frac{1}{2}V\{\ln(S_0|S_T)\} \\ &= E\left\{\ln(S_T|S_0)\right\} + \frac{1}{2}V\{\ln(S_T|S_0)\} \end{aligned} \quad (55)$$

Here the second equality follows from the inversion of the ratio. This involves a change of sign of its logarithm, which affects the expected value on the RHS but not the variance. Rearranging the expression and using the result from (54) gives

$$\left. \begin{aligned} E\{\ln(S_T|S_0)\} &= n \ln R + \frac{1}{2}V\{\ln(S_T|S_0)\} \\ &= (r + \sigma^2/2) \end{aligned} \right\} \quad (56)$$

This provides the second equality of (46).

3. NUMERICAL IMPLEMENTATION OF BINOMIAL MODEL FOR VANILLA OPTION PRICING AND NUMERICAL EXPERIMENT

This section presents the numerical implementation of binomial model for vanilla option pricing, numerical example and discussion of results as follows:

3.1 Numerical Implementation

Here we present the implementation of binomial model for pricing vanilla options as follows:

- A1. The stock price S_{ti} at ti over time step δt can only take two possible values: upward movement to $S_{ti}u$ or downward movement

to $S_{ti}d$ at t_{i+1} with $0 < d < u$ where u is the factor of upward movement and d is the factor of downward movement.

- A2. The probability of upward movement between time t_i and t_{i+1} is p and therefore the probability of downward movement is $(1 - p)$.
A3. $E(S_{t_{i+1}}|S_{t_i}) = S_{t_i}e^{r\delta t}$
A4. The probability p does not reflect the true probability of the upward movement of the underlying price of the asset. It is an artificial probability that reflects the assumption A3. From assumptions A1 and A2, we have that $E(S_{t_{i+1}}|S_{t_i}) = pS_{t_i}u + (1 - p)S_{t_i}d$. Equating this to $E(S_{t_{i+1}}|S_{t_i})$ in assumption A3 we get $S_{t_i}e^{r\delta t} = pS_{t_i}u + (1 - p)S_{t_i}d$

Therefore,

$$e^{r\delta t} = pu + (1 - p)d \quad (57)$$

And solving for p , we have that

$$p = \frac{(e^{r\delta t} - d)}{u - d} \quad (58)$$

In addition to this, at the expiry time $t = t_{M+1} = T$ there are $M + 1$ possible asset prices.

3.2 Numerical Experiment

We consider the performance of Binomial model against the “true” Black-Scholes price for American and European options with the following parameters

$$S = 45, K = 40, T = 0.5, r = 0.1, \sigma = 0.25$$

The results obtained are shown in the Table 1 below. The convergence of the binomial model to the Black-Scholes value of the option is also made more intuitive by the graph in Fig. 1 below.

Table 1. The comparative results analysis of the Cox-Ross-Rubinstein “binomial model” for pricing vanilla options to Black-Scholes values as we increase the number of steps N

N	Black-Scholes call price ($B_C = 7.6200$)		Black-Scholes put price ($B_P = 0.6692$)	
	E_C European Call	A_C American Call	E_P European Put	A_P American Put
10	7.5849	7.5849	0.6341	0.6910
30	7.6222	7.6222	0.6714	0.7258
70	7.6219	7.6219	0.6711	0.7238
120	7.6229	7.6229	0.6721	0.7238
200	7.6213	7.6213	0.6705	0.7224
270	7.6215	7.6215	0.6707	0.7223

The results given in the Table 1 above can be obtained using Matlab codes.

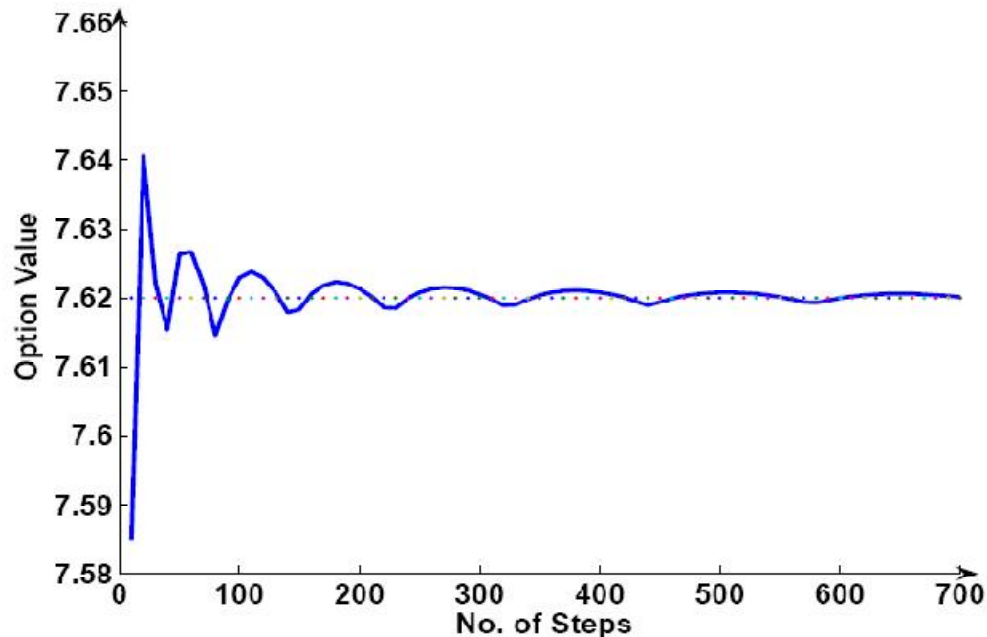


Fig. 1. Convergence of the European call price for a non-dividend paying stock using binomial model to the Black-Scholes value of 7.62

4. CONCLUSION

Options come in many different types such as path dependent and non-dependent, fixed exercise time or early exercise options and so on. Binomial model is good for dealing with some of these option flavors. The risk neutral binomial process is an alternative approach for deriving the analytic option pricing equation called "Black-Scholes Partial Differentiation Equation". Also the CRR "binomial" model has the Black-Scholes analytic formula as the limiting case as the number of steps tends to infinity.

We end this paper by commenting on the advantages of binomial model for pricing options.

4.1 Advantages of Binomial Model

- Using the numerical approach of binomial model we can calculate the American option price as well as the European option price.
- Pharmaceutical companies benefit from the use of binomial model method for real option valuation instead of older analysis as they deal with projects which have high risk and great uncertainty.
- The binomial model is much more capable of handling early exercise because it

considers the cash flow at each time period rather than just the cash flows at expiration.

4.2 Disadvantages of Binomial Model

- The binomial model is quite hard to adapt to more complex situations.
- The binomial model though can use a variant that allows the estimation of up and down movements in stock prices from the estimated variance; it cannot accurately determine what stock prices will be at the end of each period.
- Another major limitation of the binomial options pricing model is its slow speed.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Black F, Scholes M. The pricing of options and corporate liabilities. *Journal of Political Economy*. 1973;81:637-654.
2. Cox J, Ross S, Rubinstein M. Option pricing: A simplified approach. *Journal of*

3. Financial Economics. 1979;7:229-263.
4. Fadugba S, Nwozo C, Babalola T. The comparative study of finite difference method and Monte Carlo method for pricing European option. Mathematical Theory and Modeling. 2012;2:60-66.
5. Fadugba SE, Nwozo CR, Okunlola JT, Adeyemo OA, Ajayi AO. On the accuracy of binomial model and Monte Carlo method for pricing European options. International Journal of Mathematics and Statistics Studies. 2013;3:38-54.
6. Merton RC. Theory of rational option pricing. Bell Journal of Economics and Management Science. 1973;4:141-183.
7. Nwozo CR, Fadugba SE. Some numerical methods for options valuation. Communications in Mathematical Finance. 2012;1;57-74.
8. Nwozo CR, Fadugba SE. On the accuracy of binomial model for the valuation of standard options with dividend yield in the context of black-scholes model. IAENG International Journal of Applied Mathematics. 2014;44:33-44.
9. Sherwani Y. Binomial approximation methods for option pricing. U.U.D.M Project Report 2007:22. Department of Mathematics, Uppsala University; 2007.
10. Weston J, Copeland T, Shastri K. Financial theory and corporate policy. Fourth Edition, New York, Pearson Addison Wesley; 2005.
11. Wilmott P. Derivatives: The theory and practice of financial engineering. John Wiley and Sons Limited; 1998.

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