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ON MATHEMATICAL MODEL FOR THE STUDY OF TRAFFIC FLOW ON THE HIGH WAYS

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ABSTRACT

This paper presents mathematical model for the study of traffic flow on the highways. The effect of the density of cars on the overall interactions of the vehicles along a given distance of the road was investigated. We also observed that the density of cars per mile affects the net rate of interaction between them.

Keywords: Density, Highway, Mathematical model, Traffic flow, Traffic model, Road.

2010 Mathematics Subject Classification: 00A71, 81T80.

Contribution/ Originality

This study uses new estimation methodology to determine the effect of the density of cars on the overall interactions of the vehicles along a given distance of the road.

1. INTRODUCTION

Mathematical model usually describes a system by set of variables and equations that establish relationship between these variables. The values of these variables can be practically anything. Real or integer numbers, Boolean values of strings for example. These variables represent some properties of the system, for example, event occurrence (yes/no).

Road transportation is the transport of passengers or goods on roads. This is also one of the most common means of transportation by which passengers can move themselves and their goods from one station to another to achieve this, they have to use the roadways alongside with other millions of people given rise to millions of vehicles on the roadways, these vehicles interact with each other an impact overall movement of traffic which pose great challenge to the traffic-flow. In an attempt to solve these daily challenges on the roadways, a traffic flow model was developed to help the transportation engineer to understand and express the property of traffic flow on the highways.

Road transportation is still the safest means of modern movement within a city or community around the world. It is the one in which if a car is breakdown the passenger can easily highlight, change the car or walk the rest of the journey. Other means of transportations are air, canal, rail but all these do not take us to our door steps [1] .

Traffic flow is a continuous movement of vehicles along a road or street at a particular time. Traffic flow model describes a precise mathematical way of how groups of vehicles at a particular time move, interact and how their movements are being affected by density, spotlights and other infrastructures on the road ways. Whether the task is to evaluate the capacity of existing roadways or design new roadways, most transportation engineering projects begin with an evaluation of traffic flow. Therefore, the transportation engineers need to have a firm understanding of what type of flow that occurs in a given situation that will help in deciding which of the analysis method and descriptions are most relevant to model. Thus, the traffic flow can be divided into two primary types namely; Uninterrupted-Traffic-Flow and Interrupted-Traffic-Flow.

Uninterrupted traffic-flow is a flow that is regulated by vehicle-vehicle interactions and interactions between vehicles and the roadways. For instance vehicles traveling on inter-state highway are participating in an uninterrupted traffic-flow while Interrupted-traffic-flow is a flow regulated by external means, such as traffic signal. Under interrupted flow, vehicle-vehicle interactions and vehicle-roadway interactions play a secondary role in defining the traffic-flow.

The factors affecting traffic-flow provides a great opportunity to define an efficient model and mechanism to be adopted to rise up to the challenge. Traffic-flow is being affected by flow (flux), speed and density.

The scientific study of traffic flow had its beginning in the 1930s with the application of probability theory to the description of road traffic and with the pioneering studies conducted at the Yale Bureau of Highway Traffic on the study of models relating Volume and speed and the investigating of performance of traffic at intersections. At Los Alamos Labs, and General motors, engineers developed a model on Traffic flow with equations to conserve the momentum of the cars and show how the density of traffic affects the speed at which the car travels.

After World War II, with tremendous increase in the use of automobiles and the expansion of the highway system, there was also a surge in the study of traffic characteristics and the development of traffic-flow theories in December 1959, the first international symposium on the theory of Traffic flow was held at the General motors Research Laboratories in Warren, Mich. A glance through the proceedings of these symposia will provide good indication of the tremendous developments over the last 40 years in the understanding and the treatment of traffic-flow processes. Yet, even as the traffic-flow theory (from various quarters) is increasingly, better understood and more easily characterized through advanced computation technology the fundamentals are important today as in the early days. They form the foundation for all the theories, techniques, and procedures that are being applied in the design, operation and development of advanced transportation systems. There are numerous texts and literature on this subject such as ([2], [3], [4], [5], [6], [7], [8], [9], [10], [11],[12]) just to mention a few.

However, in this paper we consider the general effect of density of cars and distance over a period of time on the interaction between the vehicles.

2. MATHEMATICAL FORMULATION

Mathematical modeling is the use of a simplified mathematical representation of a real world system, process or theory. Mathematical models are developed in order to enhance our ability to understand, predict, and possibly control the behavior of the system being modeled.

Akinrelere and Ayeni [2] described mathematical modeling to be translation of physical problem using mathematical representation or equation and solves then using the result in the real problems. To achieve this aim some deduction must be made. This takes place in at least three stages namely:

- Formulate a model
- Deduction of mathematical implication within the model
- Interpretation of former steps in real term.

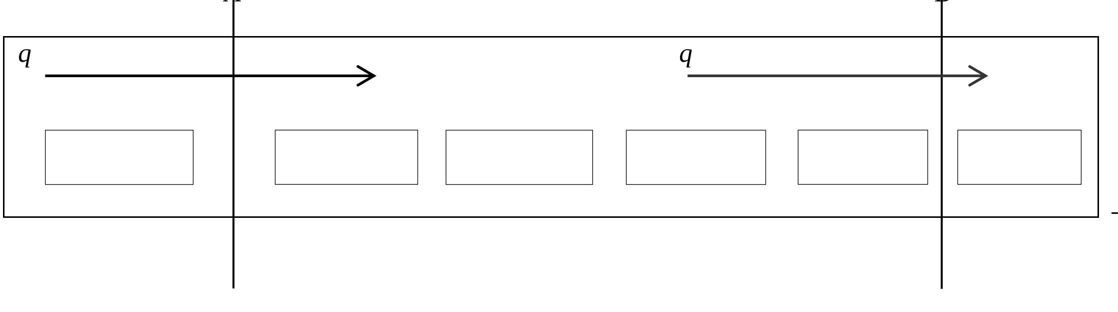
Mathematical models may also involve diagrams and graph, such as electronic or computer circuit diagrams, or stock-market index performance charts.

2.1. Modeling Traffic Flow

We consider the flow of cars on a long highway under the assumptions that cars do not enter or leave the highway at any one of the points. This road with a variable number of cars from none up to the maximum carrying capacity of the road

The amount of cars on the stretch from A to B can be modeled by taking the x-axis along the highway and assume that the traffic flows is in the positive direction. Suppose $\rho(x,t)$ is the density representing the number of cars per unit length at the point x of the highway at a time t, and $q(x,t)$ is the flow of cars per unit time. This equation is used to converse the momentum of cars and to show how the density of the traffic will affect the speed at which the cars travel

Fig-1. Traffic Flow



We assume a conservation law which states that; the change in the total amount of physical quantity contained in a region of space must be equal to the flux of the quantity across the boundary of that region. In this case, the time rate of change of the total number of cars in any segment $A \leq x \leq B$ of highway is given by

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = \frac{d}{dt}\int_{x_1}^{x_2} \frac{d\rho}{dt}dx \tag{1}$$

$$q(x_1,t) - q(x_2,t) \tag{2}$$

Equation (2) measures the flow of cars entering the segment A minus the flow of cars leaving the segment B .

This rate of change in (1) must be equal to the net flux across A and B given by (2). Thus we have the conservation equation of the form

$$\frac{d}{dt}\int_{x_2}^{x_2} \rho(x,t)dx = q(x_1,t) - q(x_2,t) \Rightarrow \int_{x_1}^{x_2} \frac{\partial \rho}{\partial t}dx = -\int_{x_1}^{x_2} \frac{\partial q}{\partial x}dx \tag{3}$$

Then (3) becomes

$$\int_{x_1}^{x_2}\left(\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x}\right)dx = 0 \tag{4}$$

Since the integrand in (3) and (4) continuous and it holds for every segment $\left(A, B\right)$, then it follows that the integrand must vanish in order to have the partial differential equation given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{5}$$

We now introduce additional assumptions which are supported by both theoretical and experimental findings:

- Measuring the velocity and position of each individual's car on the road is too difficult. So, this model views the distribution of cars by looking at the density of the cars which is the number of cars per mile on the road. We assume that the density is the only property of the cars which matters.
- The second assumption follows from the first. Only the density of the cars matters.
- Therefore the average velocity of the cars at any point depends on the density of the cars.

According to these assumptions, the flow rate q depends on x and t only through the density, that is, $q = Q(\rho)$ for some function Q this relation seems to be reasonable in the sense that the density of cars surrounding a given car indeed controls the speed limits, weather conditions, and road characteristics.

We consider here a particular relation $q = \rho v$ where v is the average local velocity of cars and we assume that (4) is a function of ρ to the first approximation, in view of this relation, (5) reduces to a nonlinear hyperbolic partial differential equation of the form

$$\frac{\partial \rho}{\partial t} + C(\rho)\frac{\partial \rho}{\partial x} = 0 \qquad (6)$$

where

$$c(\rho) = q'(\rho) = v(\rho) + \rho v'(\rho) \qquad (7)$$

In general, the local velocity $V(\rho)$ is a decreasing function of ρ, where $V(\rho)$ has a finite maximum value $V_{\max}$ at $\rho = 0$ and decrease to zero at $\rho = \rho_{\max} = \rho_m$. For the value of $\rho = \rho_m$ the cars are bumper to bumper. Since $q - \rho V, q(\rho) = 0$ when $\rho = 0$ and $\rho = \rho_m$ this means that q is an increasing function of ρ until it attains a maximum value $q_{\max} = q_m$ and some $\rho = \rho_m$ and then decreases to zero at $\rho = \rho_m$. Both $q(\rho)$ and $V(\rho)$ are shown in (i) above.

With the wave propagation velocity, (7) becomes

$$c(\rho) = V(\rho) + \rho V'(\rho) \qquad (8)$$

2.2. Equation of the Model

This section presents the equation of the model as follows:

The equation of the model is given by

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \\ &c(\rho) = q'(\rho) = v(\rho) + \rho v'(\rho) \end{aligned}\right\} \qquad (9)$$

Since $v'(\rho) < 0,\ c(\rho) < v(\rho)$

Where

$c(\rho)$ is the wave propagation velocity

$v(\rho)$ is the car velocity (miles/hour or kilo/hour)

ρ is the density (Vehicle/miles or vehicle/kilo)

t is the time (hours or seconds)

x is the distance (miles or kilometer)

q is the flux (vehicle/hour)

3. SOLUTION OF THE MODEL

Recall from (6) that

$$\frac{\partial \rho}{\partial t}+c(\rho)+\frac{\partial \rho}{\partial x}=0$$

The simplest first order non-linear wave equation is given by

$$u_t + c(u)u_x = 0, \quad -\infty < x < \infty, t > 0 \tag{10}$$

where $c(u)$ is a given function of u . Therefore let (6) be subjected to the initial condition given by

$$u(x,0) = f(x), \quad -\infty < x < \infty$$

Therefore, we can write that $\rho(x,0) = f(x), \quad -\infty < x < \infty$

Since (6) is a non-linear partial differential equation, we are going to solve the system by the method of characteristics. In order to construct continuous solutions, we consider the total derivative $d\rho$ given by $\rho(x,1)$ such that:

$$d\rho = \frac{d\rho}{dt}dt + \frac{d\rho}{dx}dx \tag{11}$$

$$\frac{d\rho}{dt} = \frac{d\rho}{dt} + \frac{d\rho}{dx}\frac{dx}{dt} \tag{12}$$

Therefore,

$$\frac{d\rho}{dt} = \rho_t + \left(\frac{dx}{dt}\right)\rho_x \tag{13}$$

It follows from this result that (6) can be regarded as the ordinary differential equation;

$$\frac{d\rho}{dt} = 0 \tag{14}$$

To assume that the point (x,t) lie on a curve Ω. then $\frac{dx}{dt}$ represents the slope of the curve at any point p on Ω. Thus along any member of the curves Ω which are the solution curves of

$$\frac{dx}{dt} = c(\rho) \tag{15}$$

These curves Ω are called the characteristics curves of the main equation (6). Thus, the solution of (6) has been reduced to the solution of a pair simultaneous ordinary differential equation (14) and (15). Equation (14) implies that ρ is constant along each characteristics curve Ω and each $c(\rho)$ remains constant on the curve. Therefore (15) shows that the characteristics of equation (1) from a family straight lines in (x,t) plane with slope $c(\rho)$ corresponds to the value of ρ on it. If the initial point on the characteristic curve Ω is denoted by ξ and if one of the curves Ω intersects, $t=0$ at $x=\xi$, then $\rho(0)=f(\xi)$

So,

$$\frac{dx}{dt}=c(\rho),\ x(0)=\xi \tag{16}$$

$$\frac{d\rho}{dt}=0,\ \rho(0)=f\left(\xi\right) \tag{17}$$

Therefore, (16) form a pair of differential equation that cannot be solved independently since *c* is a function of ρ. Equation (17) can be solved to obtain a constant ρ on Ω and hence $\rho=f(\xi)$ on the whole of Ω. Then,

$$\frac{dx}{dt}=c(f(\xi)),\ x(0)=\xi \tag{18}$$

Integrating (18) we have;

$$\int dx=\int c(f(\xi))dt$$

$$\left.\begin{array}{l} x=\xi+tc(f(\xi)) \\ \rho(x,t)=f(\xi) \end{array}\right\} \tag{19}$$

Where $F(\xi)=c(f(\xi))$

We shall verify that (19) represents an analysis expression of the solution.
Differentiating (19) w.r.t *x* and *t*, we obtain respectively

$$\rho_x=f'(\xi)\xi_x, \tag{20}$$

$$\rho_t=f'(\xi)\xi_t \tag{21}$$

$$1=(1+tf'(\xi))\xi_x$$

$$F(\xi)=(1+tF'(\xi))\xi_t$$

Eliminating ξ_x and ξ_t then (20) and (21) become respectively

$$\rho_x=\frac{f'(\xi)}{1+tF'(\xi)} \tag{22}$$

$$\rho_t=\frac{F(\xi)f'(\xi)}{1+tF'(\xi)} \tag{23}$$

Since $F(\xi)=c(f(\xi))$. Equation (6) is satisfied provided that $1+tF^1(\xi)\neq 0$. Also equation (19) satisfied the initial condition at $t=0$ since $\xi=X$, and it is unique. Suppose $\rho(x,t)$ and $q(x,t)$ are two solutions. Then on

$$\left.\begin{aligned} X &= \xi + tF(\xi) \\ \rho(x,t) &= \rho(\xi,0) = f(\xi) = q(x,t) \end{aligned}\right\} \tag{24}$$

When $c(\rho)$ is a constant and $c>0$, then (6) becomes a linear wave equation. The characteristics carves are

$$x = ct + \xi \tag{25}$$

Also the solution ρ is given by

$$\rho(x,t) = f(x-ct) \tag{26}$$

4. APPLICATION OF THE MODEL

Given that $0\le t\le 1,\ \delta t=0.1,\ 0\le x\le 100,\ \delta x=10$

Case 1

$x=100,\ c=1,\ t=0,\ \rho(x,t)=25$

$\rho(x,t)=f(x-ct)$

$x=100,\ t=0,\ c=1,\ \rho=25$

Then

$\rho(0,t)=f(x-ct)$

$\rho(x,t)=25$

$25=f(100-(1)(0))$

$25=100f$

$f=\dfrac{25}{100}$

$f=\dfrac{1}{4}$

Continuing this way we can obtain the values of f for $t=0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0$

Hence the results obtained for the values $t=0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1.0$ and $x=10,20,30,40,50,60,70,80,90,100$ using equation (26) are shown in Table 1.

Case 2

$x = 100, t = 0, c = 1, \rho(x,t) = 50$

$\rho = f(x - ct) = 50$

$50 = f(100 - (1)(0))$

$50 = 100f$

$f = \frac{1}{2}$

Continuing this way we can obtain the values of f for

$t = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$

Hence the results obtained for the values $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ and $x = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$ using equation (26) are shown in Table 2.

Given that $0 \le t \le 1, \delta t = 0.1, 0 \le x \le 50, \delta x = 10$

Case 3

$x = 50, t = 0, c = 1, \rho(x,t) = 25$

$\rho(x,t) = f(x - ct)$

$25 = f(50 - 0)$

$25 = 50f$

$f = \frac{25}{50}$

$f = \frac{1}{2}$

Continuing this way we can obtain the values of f for

$t = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$

Hence the results obtained for the values $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ and $x = 10, 20, 30, 40, 50$ using equation (26) are shown in Table 3.

Case 4

$x = 50, t = 0, c = 1, \rho(x,t) = 50$

$\rho(x,t) = f(x - ct)$

$50 = f(50 - (1)(0))$

$50 = f(50 - 0)$

$50 = 50f$

$f = 1$

Continuing this way we can obtain the values of f for

$t = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$

Hence the results obtained for the values $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ and $x = 10, 20, 30, 40, 50$ using equation (26) are shown in Table 4.

4.1. Discussion of Results

Tables 1, 2, 3 and 4 show that as the density of cars increases along a given length of road, the rate of interaction between the cars increases along the highways. Similarly, we observed that as the cars moves along the road from one point to another the rate of interaction also increases.

Conversely, as the time progresses at a given distance, we observed that the rate of interaction reduces.

4.2. Table of Results

The results generated are shown in the Tables below as follows:

Table-1. $\rho = 25, 0 \le x \le 100, 0 \le t \le 1, c = 1$

t/x	0	10	20	30	40	50	60	70	80	90	100
0.0	0.000	2.500	5.000	7.500	10.00	12.500	15.000	17.500	20.000	22.500	25.000
0.1	0.025	2.475	4.975	7.475	9.975	12.475	14.975	17.475	19.957	22.475	24.475
0.2	0.050	2.450	4.950	7.450	9.950	12.450	14.950	17.450	19.950	22.450	24.950
0.3	0.075	2.425	4.925	7.425	9.925	12.425	14.925	17.425	19.925	22.425	24.925
0.4	0.100	2.400	4.900	7.400	9.900	12.400	14.900	17.400	19.900	22.400	24.900
0.5	0.125	2.375	4.875	7.375	9.875	12.375	14.875	17.375	19.875	22.375	24.875
0.6	0.150	2.350	4.850	7.350	9.850	12.350	14.850	17.350	19.850	22.350	24.850
0.7	0.175	2.325	4.825	7.325	9.825	12.325	14.825	17.325	19.825	22.325	24.825
0.8	0.200	2.300	4.800	7.300	9.800	12.300	14.800	17.300	19.800	22.300	24.800
0.9	0.225	2.275	4.775	7.275	9.775	12.275	14.775	17.275	19.775	22.775	24.775
1.0	0.250	2.250	4.750	7.250	9.750	12.250	14.750	17.250	19.750	22.500	24.750

Table-2. $\rho = 50, 0 \le x \le 100, 0 \le t \le 1, c = 1$

t/x	0	10	20	30	40	50	60	70	80	90	100
0.0	0.00	5.00	10.0	15.00	20.00	25.00	30.00	35.00	40.00	45.00	50.00
0.1	0.05	4.95	9.95	14.95	19.95	24.95	29.95	34.95	39.95	44.95	49.95
0.2	0.10	4.90	9.90	14.90	19.90	24.90	29.90	34.90	39.90	44.90	49.90
0.3	0.15	4.85	9.85	14.85	19.85	24.85	29.85	34.85	39.85	44.85	49.85
0.4	0.20	4.80	9.80	14.80	19.80	24.80	29.80	34.80	39.80	44.80	49.80
0.5	0.25	4.75	9.75	14.75	19.75	24.75	29.75	34.75	39.75	44.75	49.75
0.6	0.30	4.70	9.70	14.70	19.70	24.70	29.70	34.70	39.70	44.70	49.70
0.7	0.35	4.65	9.65	14.65	19.65	24.65	29.65	34.65	39.65	44.65	49.65
0.8	0.40	4.60	9.60	14.60	19.60	24.60	29.60	34.60	39.60	44.60	49.60
0.9	0.45	4.55	9.55	14.55	19.55	24.55	29.55	34.55	39.55	44.55	49.55
1.0	0.50	4.50	9.50	14.50	19.50	24.50	29.50	34.50	39.50	44.50	49.50

Table-3. $\rho(x,t) = 25,\ 0 \le x \le 50,\ 0 \le t \le 1,\ c = 1$

t/x	0	10	20	30	40	50
0.0	0.00	5.00	10.0	15.00	20.00	25.00
0.1	0.05	4.95	9.95	14.95	19.95	24.95
0.2	0.10	4.90	9.90	14.90	19.90	24.90
0.3	0.15	4.85	9.85	14.85	19.85	24.85
0.4	0.20	4.80	9.80	14.80	19.80	24.80
0.5	0.25	4.75	9.75	14.75	19.75	24.75
0.6	0.30	4.70	9.70	14.70	19.70	24.70
0.7	0.35	4.65	9.65	14.65	19.65	24.65
0.8	0.40	4.60	9.60	14.60	19.60	24.60
0.9	0.45	4.55	9.55	14.55	19.55	24.55
1.0	0.50	4.50	9.50	14.50	19.50	24.50

Table-4. $\rho(x,t) = 50,\ 0 \le x \le 50,\ 0 \le t \le 1,\ c = 1$

t/x	0	10	20	30	40	50
0.0	0.0	10	20.0	30.0	40.0	50.0
0.1	0.1	9.9	19.9	29.9	39.9	49.9
0.2	0.2	9.8	19.8	29.8	39.8	49.8
0.3	0.3	9.7	19.7	29.7	39.7	49.7
0.4	0.4	9.6	19.6	29.6	39.6	49.6
0.5	0.5	9.5	19.5	29.5	39.5	49.5
0.6	0.6	9.4	19.4	29.4	39.4	49.4
0.7	0.7	9.3	19.3	29.3	39.3	49.3
0.8	0.8	9.2	19.2	29.2	39.2	49.2
0.9	0.9	9.1	19.1	29.1	39.1	49.1
1.0	1.0	9.0	19.0	29.0	39.0	49.0

5. CONCLUSION

As earlier mentioned this model is a precise mathematical description of how groups of moving vehicles interact and being affected by density, spotlights and other infrastructures on the roadways. In view of this definition it is clear that traffic-flow model is an indispensable tool to assist and enable the traffic and transportation engineers to understand and express the properties of traffic-flow on our roadways so as to design diligent, effective and operational street and highways thus making the roadways easier to drive on.

The model also assists engineers to know where to put spotlights and other infrastructures like signage, control devices, markings etc. to reduce traffic congestion on the highways.

The model directly or indirectly contributes to the improvement of accidents free on roadways by providing suitable mechanism to the construction of roadways. It gives high fortune to the maintenance of vehicles. Traffic-flow model provides traffic-engineers with the means to evaluate system wide control strategies in urban areas

The quality of service provided to motorists can be monitored to assess the city’s ability to manage growth.

The model could also be used to compare traffic conditions among different cities in order to determine the allocation of resources for transportation system improvement.

This model could be used to explain vehicle to vehicle interaction on the roadways and how highway infrastructures affect their movement which helps the traffic engineers to make the roads easier to drive on and suggest the appropriate number (density) of cars that should plight a particular road at a given time.

REFERENCES

[1] O. G. Omogunloye and O. O. Ayeni, "An application of the average delay model to traffic congestion at a multi road junction point," *Research Journal in Engineering and Applied Sciences,* vol. 1, pp. 342-350, 2012.

[2] E. A. Akinrelere and R. O. Ayeni, "Shockwaves in arterial blood flow," *Journal of Nigerian Mathematical Society,* vol. 2, pp. 91 - 98, 1983.

[3] B. D. Greenshields, J. R. Bibbins, W. S. Channing, and H. H. Miller, "A study of traffic capacity," *Highway Research Board Proceedings*, vol. 14, pp. 448-477, 1935.

[4] R. Herman, E. W. Montroll, R. Potts, and R. W. Rothery, "Traffic dynamics: Analysis of stability in car-following," *Operations Research,* vol. 1, pp. 86-106, 1959.

[5] L. D. Landau and E. M. Lifshitz, *Fluid mechanics*, 2nd ed. vol. 6: Butterworth-Heinemann, 1987.

[6] H. L. Liu and Z. Wang, "The discontinuous finite element method for red-and- green light models for the traffic flow," *Mathematics and Computers in Simulation,* vol. 2009, pp. 07-16, 2001.

[7] R. E. Mickens, *Non-standard finite-difference models of differential equations*. Singapore: World Scientific, 1994.

[8] J. H. Olseen and A. Sharpiro, "Large amplitude unsteady flow in liquid filled elastic tubes," *J. Fluid Mechanics*, vol. 29, pp. 513-538, 1967.

[9] P. I. Richards, "Shock waves on the highway," *Operations Research,* vol. 4, pp. 42-51, 1956.

[10] H. G. Nathan, J. M. Carroll, and K. R. Ajay, *Monograph on traffic flow theory; US department of transportation* vol. 62: Federal Highway Administration Pub, 1997.

[11] W. R. Reilly, C. C. Gardner, and J. H. Kell, "A technique for measurement of delay at intersections," FHWA Report RD-76-135/137, FHWA, U.S. Department of Transportation, Washington, D.C.1976.

[12] Transportation Research Board, "Providing a highway system with reliable travel times," *Future Strategic Highway Research Plan Area 3. Available* *http://wwwcf.fhwa.dot.gov/exit.cfm?link=http://www4.trb.org/trb/newshrp.nsf/web/progress_reports?OpenDocument*, 2003.