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ON THE STRENGTH AND WEAKNESS OF BINOMIAL MODEL FOR PRICING VANILLA OPTIONS

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Abstract: *This paper presents binomial model for pricing vanilla options. Binomial model can be used to accurately price American style options than the Black-Scholes model as it takes into consideration the possibilities of early exercise and other factors like dividends. The strength and weakness of this model were considered. This model is both computationally efficient and accurate but not adequate to deal with path dependent options.*

Keywords: *American Option, Binomial Model, Black Scholes Model, European Option, Vanilla option.*

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1.0 INTRODUCTION

In the past two decades, options have undergone a transformation from specialized and obscure securities to ubiquitous components of the portfolio of not only large fund managers, but also ordinary investors. Essential ingredients of any successful modern investment strategy include the ability to generate income streams and reduce risk, as well as some level of speculation all of which can be accomplished by effective use of options.

An option is a financial contract or a contingent claim that gives the holder the right, but not the obligation to buy or sell an underlying asset for a predetermined price called the strike or exercise price during a certain period of time. Options come in a variety of "flavours". A vanilla option offers the right to buy or sell an underlying security by a certain date at a set strike price. In comparison to other option structures, vanilla options are not fancy or complicated. Such options may be well-known in the markets and easy to trade. Increasingly, however, the term vanilla option is a relative measure of complexity, especially when investors are considering various options and structures. Examples of vanilla options are an American option which allows exercise at any point during the life of the option and a European option that allows exercise to occur only at expiration.

Black and Scholes published their seminar work on option pricing [1] in which they described a mathematical frame work for finding the fair price of a European option. They used a no-arbitrage argument to describe a partial differential equation which governs the evolution of the option price with respect to the maturity time and the price of the underlying asset.

The subject of numerical methods in the area of option pricing and hedging is very broad, putting more demands on computation speed and efficiency. A wide range of different types of contracts are available and in many cases there are several candidate models for the stochastic evolution of the underlying state variables [10].

Now, we present an overview of binomial model in the context of Black-Scholes-Merton [1, 8] for pricing vanilla options based on a risk-neutral valuation which was first suggested and derived by Cox-Ross-Rubinstein [4] and assumes that stock price movements are composed of a large number of small binomial movements. Other procedures are finite difference methods for pricing derivative governed by solving the underlying partial differential equations was considered by Brennan and Schwarz [3] and Monte Carlo method for pricing European option and path dependent options was introduced by Boyle [2]. The comparative study of finite difference method and Monte Carlo method for pricing European option was considered by Fadugba, Nwozo and Babalola [6]. Later, on the stability and accuracy of finite difference method for option pricing was considered by Fadugba and et al [5]. These procedures provide much of the infrastructure in which many contributions to the field over the past three decades have been centered.

In this paper we shall consider only the strength and weakness of binomial model for pricing vanilla options namely American and European options.

2.0 BINOMIAL MODEL

This is defined as an iterative solution that models the price evolution over the whole option validity period. For some vanilla options such as American option, iterative model is the only choice since there is no known closed form solution that predicts its price over a period of time. The Cox-Ross-Rubinstein “Binomial” model [4] contains the Black-Scholes analytic formula as the limiting case as the number of steps tends to infinity. Next we shall present the derivation and the implementation of the binomial model below.

2.1 THE COX-ROSS-RUBINSTEIN MODEL [4, 7]

We know that after a period of time, the stock price can move up to Su with probability p or down to Sd with probability $(1-p)$, where $u>1$ and $0<d<1$. Therefore the corresponding value of the call option at the first time movement δt is given by

$$f_u = \max(Su - K, 0) \tag{1}$$

$$f_d = \max(S_d - K, 0) \tag{2}$$

Where f_u and f_d are the values of the call option after upward and downward movements respectively.

We need to derive a formula to calculate the fair price of vanilla options. The risk neutral call option price at the present time is given by

$$f = e^{-r\delta t}[pf_u + (1-p)f_d] \tag{3}$$

Where the risk neutral probability is given by

$$p = \frac{e^{r\delta t} - d}{u - d} \tag{4}$$

Now, we extend the binomial model to two periods. Let f_{uu} denote the call value at time $2\delta t$ for two consecutive upward stock movements, f_{ud} for one downward and one upward movement and f_{dd} for two consecutive downward movements of the stock price [9]. Then we have

$$f_{uu} = \max(Suu - K, 0)$$

(5)

$$f_{ud} = \max(Sud - K, 0)$$

(6)

$$f_{dd} = \max(Sdd - K, 0)$$

(7)

The values of the call options at time δt are

$$f_u = e^{-r\delta t}[pf_{uu} + (1-p) f_{ud}] \tag{8}$$

$$f_d = e^{-r\delta t}[pf_{ud} + (1-p) f_{dd}] \tag{9}$$

Substituting (8) and (9) into (3), we have

$$f = e^{-r\delta t}[pe^{-r\delta t} f_{uu} + (1-p) f_{ud} + (1-p)e^{-r\delta t}(pf_{ud} + (1-p) f_{dd})]$$

$$f = e^{-2r\delta t}[p^2 f_{uu} + 2p(1-p) f_{ud} + (1-p)^2 f_{dd})] \tag{10}$$

Equation (10) is called the current call value, where the numbers p^2, $2p(1-p)$ and $(1-p)^2$ are the risk neutral probabilities for the underlying asset prices Suu, Sud and Sdd respectively.

We generalize the result in (10) to value an option at $T = N\delta t$ as follows

$$f = e^{-Nr\delta t} \sum_{j=0}^{N} {}^N C_j p^j (1-p)^{N-j} f_{u^j d^{N-j}}$$

$$f = e^{-Nr\delta t} \sum_{j=0}^{N} {}^N C_j p^j (1-p)^{N-j} \max(Su^j d^{N-j} - K, 0) \tag{11}$$

Where $f_{u^j d^{N-j}} = \max(Su^j d^{N-j} - K, 0)$ and ${}^N C_j = \frac{N!}{(N-j)!\, j!}$ is the binomial coefficient. We assume that m is the smallest integer for which the option's intrinsic value in (11) is greater than zero. This implies that $Su^m d^{N-m} \geq K$. Then (11) can be written as

$$f = Se^{-Nr\delta t}\sum_{j=0}^{N} {}^{N}C_j p^j (1-p)^{N-j} u^j d^{N-j}$$

$$- Ke^{-Nr\delta t}\sum_{j=0}^{N} {}^{N}C_j p^j (1-p)^{N-j} f_{u^j d^{N-j}} \qquad (12)$$

Equation (12) gives us the present value of the call option.

The term $e^{-Nr\delta t}$ is the discounting factor that reduces f to its present value. We can see from the first term of (12) that -1is the binomial probability of j upward movements to occur after the first N trading periods and $Su^j d^{N-j}$ is the corresponding value of the asset after j upward movements of the stock price. The second term of (12) is the present value of the option's strike price. Let $Q = e^{-r\delta t}$, we substitute Q in the first term of (12) to yield

$$f = SQ^{-N}\sum_{j=0}^{N} {}^{N}C_j p^j (1-p)^{N-j} u^j d^{N-j}$$

$$- Ke^{-Nr\delta t}\sum_{j=0}^{N} {}^{N}C_j p^j (1-p)^{N-j} f_{u^j d^{N-j}}$$

$$f = S\sum_{j=0}^{N} {}^{N}C_j [Q^{-1}pu]^j [Q^{-1}(1-p)d]^{N-j}$$

$$- Ke^{-Nr\delta t}\sum_{j=0}^{N} {}^{N}C_j p^j (1-p)^{N-j} f_{u^j d^{N-j}} \qquad (13)$$

Now, let $\Phi(m; N, p)$ be the binomial distribution function given by

$$\Phi(m; N, p) = \sum_{j=0}^{N} {}^{N}C_j p^j (1-p)^{N-j} \qquad (14)$$

Equation (14) is the probability of at least m success in N independent trials, each resulting in a success with probability p and in a failure with probability $(1-p)$. Then let $p' = Q^{-1}pu$ and $(1-p') = Q^{-1}(1-p)d$. Consequently, it follows that

$$f = S\Phi(m; N, p') - Ke^{-rT}\Phi(m; N, p) \qquad (15)$$

The model in (15) was developed by Cox-Ross-Rubinstein [6], where $\delta t = \frac{T}{N}$ and we will refer to it as CRR model. The corresponding put value of the European option can be obtained using call put relationship of the form $C_E + Ke^{-rt} = P_E + S$ as

$$f = Ke^{-rT}\Phi(m;N,p) - S\Phi(m;N,p') \qquad (16)$$

Where the risk free interest rate is denoted by r, C_E is the European call, P_E is the European put and S is the initial stock price. European option can only be exercised at expiration, while for an American option, we check at each node to see whether early exercise is advisable to holding the option for a further time period δt. When early exercise is taken into consideration, the fair price must be compared with the option's intrinsic value [7].

2.2 NUMERICAL IMPLEMENTATION

Now, we present the implementation of binomial model for pricing vanilla options as follows.

When stock price movements are governed by a multi-step binomial tree, we can treat each binomial step separately. The multi-step binomial tree can be used for the American and European style options.

Like the Black-Scholes, the CRR formula in (15) can only be used in the valuation of European style options and can easily be implemented in Matlab. To overcome this problem, we use a different multi-period binomial model for the American style options on both the dividend and non-dividend paying stocks. Now we present the Matlab implementation.

The stock price of the underlying asset for non-dividend and dividend paying stocks are given respectively by

$$Su^{j}d^{N-j},\ j = 0,1,\ldots,N,\ N = 0,1,\ldots i-1 \qquad (17)$$

$$S(1-\lambda)u^{j}d^{N-j},\ j = 0,1,\ldots,N,\ N = i, i+1,\ldots \qquad (18)$$

Where the dividend is denoted by λ that reduces underlying price of the asset.

For the European call and put options, the Matlab code takes into consideration on the prices at the maturity date T and the stock prices for non-dividend paying stocks in (17). The call and put prices of European option are given by (15) and (16) respectively.

For the American call and put options, the Matlab code will incorporate the early exercise privilege and the date T, when the dividend will be paid. Then, it implies that the stock prices will exhibit (17) and (18). The call and put prices of American option for non-dividend paying stock are given by

$$f = \max[S_T - K, (S\Phi(m; N, p') - Ke^{-rT}\Phi(m; N, p))] \quad (19)$$

$$f = \max[K - S_T, (Ke^{-rT}\Phi(m; N, p) - S\Phi(m; N, p'))]$$

(20)

For dividend paying stock, we replace (17) with (18) in (12) and substitute in (19) and (20) to get respectively the call and put prices of American option.

3.0 NUMERICAL EXAMPLES

Now, we present some numerical examples.

Example 1

We compute the values of vanilla options. The results in Tables 1 and 2 for both European and American options are compared to those obtained using Black-Scholes analytic pricing formula. The rate of convergence for binomial model may be assessed by repeatedly doubling the number of time step N. Tables 1 and 2 use the parameters below in computing the options prices as we increase the number of steps.

$$S = 45, K = 40, T = \frac{1}{2}, r = \frac{1}{10}, \sigma = \frac{1}{4}$$

The Black-Scholes price for call and put options are 7.6200 and 0.6692 respectively.

Table 1: Comparison of the Binomial Model to Black-Scholes Value of the Option as we increase N

N	European call	American call	European put	American put
10	7.6184	7.6184	0.6676	0.7124
20	7.6305	7.6305	0.6797	0.7235
30	7.6042	7.6042	0.6534	0.7027
40	7.6241	7.6251	0.6742	0.7228
50	7.6070	7.6070	0.6562	0.7101
60	7.6219	7.6219	0.6710	0.7199
70	7.6209	7.6209	0.6701	0.7207
80	7.6124	7.6124	0.6616	0.7134
90	7.6210	7.6210	0.6702	0.7201
100	7.6216	7.6216	0.6707	0.7214

Table 2: The Comparison of the Convergence of the Binomial Model and Black-Scholes Value of Option as we double the value of N

N	European Call	American Call	European Put	American Put
20	7.6305	7.6305	0.6797	0.7235
40	7.6251	7.6251	0.6742	0.7228
60	7.6219	7.6219	0.6710	0.7199
80	7.6124	7.6124	0.6616	0.7134
100	7.6216	7.6216	0.6707	0.7214
120	7.6181	7.6181	0.6673	0.7182
140	7.6209	7.6209	0.6700	0.7211
160	7.6178	7.6178	0.6670	0.7184
180	7.6211	7.6211	0.6703	0.7213
200	7.6171	7.6171	0.6663	0.7185

Example 2

Consider pricing a vanilla option on a stock paying a known dividend yield with the following parameters:

$$S = 50, r = \frac{1}{10}, T = \frac{1}{2}, \sigma = \frac{1}{4}, \tau = \frac{1}{6}, \lambda = \frac{1}{20}$$

Table 3: Out of the Money, at the Money and in the Money Vanilla Options on a Stock Paying a Known Dividend Yield

K	European Call	American Call	Early Exercise Premium	European Put	American Put	Early Exercise Premium
30	18.97	20.50	1.53	0.004	0.004	0.00
45	6.06	6.47	0.41	1.37	1.49	0.12
50	3.32	3.42	0.10	3.38	3.78	0.40
55	1.62	1.63	0.01	6.40	7.31	0.91
70	0.11	0.11	0.00	19.19	21.35	2.16

4.0 DISCUSSION OF RESULTS

We can see from Table 1 that Black-Scholes formula for the European call option can be used to value its counterpart American call option for it is never optimal to exercise an American call option before expiration. As we increase the value of N , the value of the American put option is higher than the corresponding European put option as we can see from the above Tables because of the early exercise premium. Sometime the early exercise of the American put option can be optimal.

Table 2 shows that binomial model converges faster and closer to the Black-Scholes value as the value of N is doubled. This method is very flexible in pricing vanilla option.

Table 3 shows that the American option on the dividend paying stock is always worth more than its European counterpart. A very deep in the money, American option has a high early exercise premium. The premium of both put and call option decreases as the option goes out of the money. The American and European call options are not worth the same as it is optimal to exercise the American call early on a dividend paying stock. A deep out of the money, American and European call options are worth the same. This is due to the fact that they might not be exercised early as they are worthless. The above results can be obtained using Matlab codes.

5.0 CONCLUSION

Options come in many different flavours such as path dependent or non-path dependent, fixed exercise time or early exercise options and so on. Binomial model is suited to dealing with some of these option flavours.

In general, binomial model has its strengths and weaknesses of use. This model is good for pricing options with early exercise opportunities, accurate, converges faster and it is relatively easy to implement but can be quite hard to adapt to more complex situations.

We conclude that binomial model is good for pricing vanilla options most especially American and European options.

REFERENCES

[1] Black F. and Scholes M. (1973): The Pricing of Options and Corporate Liabilities, Journal of Political Economy, Vol. 81(3), 637-654.

[2] Boyle P. (1977): Options: A Monte Carlo Approach, Journal of Financial Economics. Vol. 4(3), 323-338

[3] Brennan M. and Schwartz E. (1978): Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims, Journal of Financial and Quantitative Analysis, Vol. 5(4), 461-474.

[4] Cox J., Ross S. and Rubinstein M. (1979): Option Pricing: A Simplified Approach, Journal of Financial Economics, Vol. 7, 229-263.

[5] Fadugba S. E. and et al. (2012): On the Stability and Accuracy of Finite Difference Method for Option Pricing, International Institute of Science, Technology and Education, Journal of Mathematical Theory and Modeling, Vol. 2, No. 6, 101-108, USA.

[6] Fadugba S., Nwozo C. and Babalola T. (2012): The Comparative Study of Finite Difference Method and Monte Carlo Method for Pricing European Option, International Institute of Science, Technology and Education, Journal of Mathematical Theory and Modeling, Vol. 2, No. 4, 60-66, USA.

[7] Hull J. (2003): Options, Futures and other Derivatives. Pearson Education Inc. Fifth Edition, Prentice Hall, New Jersey.

[8] Merton R. C. (1973): Theory of Rational Option Pricing. Bell Journal of Economics and Management Science, Vol. 4(1), 141-183.

[9] Ross S.M. (1999): An Introduction to the Mathematical Finance: Options and Other Topics. Cambridge University Press, Cambridge.

[10] Weston J., Copeland T. and Shastri K. (2005): Financial Theory and Corporate Policy. Fourth Edition, New York, Pearson Addison Wesley.